Quiz 1

Fair: \( H : \frac{1}{2} \) \( T : \frac{1}{2} \) \[ EF \] 80%
Defective \( H = \frac{9}{10} \) \( T = \frac{1}{10} \) \[ D \] 20%

Test: Fail if \( HH \), Pass [P] otherwise

\[ a \]
\[ P(\text{FAIL} | D) = P(\text{HH} | D) = 0.9 \cdot 0.9 = 81\% \]

\[ b \]
\[ P(\text{PASS} | F) = 1 - P(\text{HH} | F) = 1 - 0.5 \cdot 0.5 = 75\% \]

\[ c \]
\[ P(\text{D} | \text{PASS}) = \frac{P(\text{D}) P(\text{PASS} | D)}{P(\text{D}) P(\text{PASS} | D) + P(F) P(\text{PASS} | F)} \]

Bayes

\[ \frac{1 - 0.81}{0.2 \cdot 0.19} \]

\[ \frac{0.2 \cdot 0.19}{0.2 \cdot 0.19 + 0.8 \cdot 0.75} \approx 5.96\% \]
INDEPENDENCE

Events $A$, $B$ are independent if:

1. $P(A | B) = P(A)$
2. $P(B | A) = P(B)$
3. $P(A \cap B) = P(A) \cdot P(B)$

These 3 definitions are equivalent if $P(A), P(B) \neq 0$. 

\[
P(A | B) = \frac{P(A \cap B)}{P(B)}
\]

\[
P(B | A) = \frac{P(A \cap B)}{P(A)}
\]

\[
P(A \cap B) = P(A) \cdot P(B)
\]
EXAMPLE: We roll two fair 4-sided dice.

Event A: 1st roll is 1
Event B: Sum is 3

\[ P(A) = \frac{1}{4}, \quad P(B) = \frac{2}{16} = \frac{1}{8} \]

\[ P(A \cap B) = \frac{1}{16} > \frac{1}{4} \cdot \frac{1}{8} = P(A) \cdot P(B) \]

\[ P(B \mid A) = \frac{1}{4} > \frac{1}{8} = P(B) \]

Event C: Sum is 5

Are A, C independent?

\[ P(C) = \frac{4}{16} = \frac{1}{4} \quad P(C \mid A) = \frac{1}{4} \]

\[ P(A \cap C) = \frac{1}{16} = \frac{1}{4} \cdot \frac{1}{4} = P(A) \cdot P(C) \]

\[ \Rightarrow \text{YES INDEPENDENT} \]
\[ A: \text{minimum is } 2 \]
\[ B: \text{maximum is } 2 \]
\[ P(A) = \frac{5}{16}; \quad P(B) = \frac{3}{16} \]
\[ P(ANB) = \frac{1}{16} \quad \Rightarrow \quad P(A) \cdot P(B) = \frac{5}{16} \cdot \frac{3}{16} = \frac{15}{16} \]

\[ \Rightarrow \text{Not independent} \]

Independence of more than two events

\begin{align*}
\text{If } A_1, A_2, \ldots, A_n \text{ are independent, then for all } i, j \text{, and } i \neq j, \\
P(A_i A_j A_{\neq i}) &= P(A_i) P(A_j) \quad \text{and} \quad P(A_i A_{\neq i} A_{\neq j}) = P(A_i) P(A_{\neq i}) P(A_{\neq j})
\end{align*}

\[ \text{For any } \emptyset \neq I \subseteq \{ 1, \ldots, n \}: \quad P(\bigcap_{i \in I} A_i) = \prod_{i \in I} P(A_i) \]
3 events $A, B, C$ are independent if:

\[
\begin{align*}
P(A \cap B) &= P(A) \cdot P(B) \\
P(A \cap C) &= P(A) \cdot P(C) \\
P(B \cap C) &= P(B) \cdot P(C) \\
P(A \cap B \cap C) &= P(A) \cdot P(B) \cdot P(C)
\end{align*}
\]

Flip two fair coins:

- $H_1$: 1st flip $H$
- $H_2$: 2nd flip $H$
- $D$: two flips are different

\[
P(H_1) = P(H_2) = P(D) = \frac{1}{2}
\]

\[
P(H_1 \cap H_2 \cap D) = 0 + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}
\]

$H_1$, $H_2$, $D$ independent

\[
P(H_1 \cap D) = \frac{1}{4} = P(H_1) \cdot P(D)
\]

\[
\sum \text{same, } 1 \epsilon 2
\]
Roll 2 fair dice

A: 1st roll is 1, 2, or 3 \[ P(A) = \frac{1}{2} \]
B: 1st roll is 3, 4, or 5 \[ P(B) = \frac{1}{2} \]

C: Sum is 9 \[ P(C) = \frac{4}{36} = \frac{1}{9} \]

\[ P(A \cap B \cap C) = \frac{1}{36} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{9} = P(A) \cdot P(B) \cdot P(C) \]

A, B not independent: \[ P(A \cap B) = \frac{1}{6} \neq \frac{1}{2} \cdot \frac{1}{2} = P(A) \cdot P(B) \]

A, C not independent: \[ P(A \cap C) = \frac{1}{6} \neq \frac{1}{2} \cdot \frac{1}{3} = P(A) \cdot P(C) \]

B, C not independent:

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Let \( A_1, \ldots, A_n \) independent, then

\[ P(A_j \mid \bigcap_{i \neq j} A_i) = P(A_j) \quad j \neq i \]

\( A_1, B \) independent, then \( A^c, B; A, B^c; A^c, B^c \) are independent.
Conditional independence

Events $A, B$ are conditionally independent given event $C$ if

$$P(A \cap B \mid C) = P(A \mid C) \cdot P(B \mid C)$$

**Example**  Two coin flips [FA12]

$H_1, H_2$ independent, but not conditionally independent given $D$

$$P(H_1 \mid D) = \frac{1}{2}, \quad P(H_1 \cap H_2 \mid D) = 0 \neq \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$P(H_2 \mid D) = \frac{1}{2}$

Independence

Conditional independence

**Example**  Two coins

$A$: H 90%, T 10%  We choose a coin at random $[P(A) = P(B) = \frac{1}{2}]$

$B$: H 10%, T 90%

$H_1, H_2$ are conditionally independent given $A [\lor B]$
\[ P(H_1 | A) = 0.99 \quad P(H_1 H_2 | A) = 0.99 \cdot 0.01 \]
\[ P(H_2 | A) = 0.01 \]

\[ H_1, H_2 \text{ not independent} \]

\[ P(H_1) = P(A) P(H_1 | A) + P(B) P(H_1 | B) = 0.5 \cdot 0.99 + 0.5 \cdot 0.01 = 0.5 \]
\[ P(H_2) = P(A) P(H_2 | A) + P(B) P(H_2 | B) = 0.5 \cdot 0.01 + 0.5 \cdot 0.99 = 0.5 \]

\[ P(H_1 H_2) = P(A) P(H_1 H_2 | A) + P(B) P(H_1 H_2 | B) = 0.49 \]

\[ 0.5 \cdot 0.99 \cdot 0.99 \quad 0.5 \cdot 0.01 \cdot 0.01 \]

\[ P(H_1 H_2) = 0.49 > 0.5 \cdot 0.05 = 0.25 \leq P(H_1) P(H_2) \]