Prob. space: \((\Omega, \mathcal{A}, P)\)

- \(\Omega\): "sample space"
- \(\mathcal{A}\): set of "events"
- \(P: \mathcal{A} \rightarrow \mathbb{R}\)

\[ P(A) \] "probability of event \(A\)

\[ A \cup B \leftrightarrow A \lor B \]

\[ A \cap B \leftrightarrow A \land B \]

\[ A^c = \Omega \setminus A \leftrightarrow \text{NOT } A \]

\[ A \subseteq B \leftrightarrow A \Rightarrow B \] "A implies B"
Axioms and Properties of Probability

1. \(P(A) \geq 0\)
2. \(P(\emptyset) = 0\)
3a. \(P(A \cup B) = P(A) + P(B)\)
3b. \(P\left(\bigcup_i A_i\right) = \sum P(A_i)\)
4. \(\Omega = A \cup \overline{A}\)
5. \(ACB \Rightarrow P(A) \leq P(B)\)
6. \(P(A) \in [0, 1]\)
7. \(P\left(\bigcup_i A_i\right) = \sum_{i=1}^{\infty} P(A_i)\)
8. \(P(A \cup B) = P(A) + P(B) - P(ANB)\) (\(\times\))
9. \(P(A^c) = 1 - P(A)\)
10. \(\overline{A} = \Omega \setminus A\)

\(\Omega = A \cup \overline{A}\)
Classical probability - \((\Omega, \mathcal{A}, P)\)

- \(|\Omega| < \infty\) finite # of poss. outcomes, all equally likely
- \(\mathcal{A} = 2^\Omega \cup \{\emptyset\}\) [all subsets of \(\Omega\)] \(A \in \mathcal{A} \Rightarrow A \subset \Omega\)
- \(A \in \Omega\) \(\Rightarrow P(A) = \frac{|A|}{|\Omega|} \Rightarrow \#\) of all possible outcomes

\(P(A) = \frac{|A|}{|\Omega|} \Rightarrow \#\) of "favorable" outcomes

\(\text{Ex}\) dice roll: \(P(\text{we roll at least 5}) = \frac{|\{5, 6\}|}{|\{1, \ldots, 6\}|} = \frac{2}{6} = \frac{1}{3}\)

Classical prob satisfies axioms 1-3

Interpretations of probability:

- **Frequentist**: We repeat an experiment \(n\) times, event \(A\) happens \(K\) times \(\Rightarrow \frac{K}{n} \Rightarrow P(A) \sim \frac{K}{n}\)
- **Subjective**: \(P(A)\) is our degree of belief in event \(A\)
Odds for an event
odds of event A: "3:1 against"

\[
P(A) = \frac{b}{a+b}
\]

\begin{array}{cccc}
\text{Bet} & \text{Win} & \text{Lose} & P \\
1 & 1 + 3 & 0 & \frac{1}{1+3} = \frac{1}{4} \\
1 & 1 + 1 & 0 & \frac{1}{1+1} = \frac{1}{2}
\end{array}
Example

<table>
<thead>
<tr>
<th>Odds</th>
<th>Bet</th>
<th>Win</th>
</tr>
</thead>
<tbody>
<tr>
<td>2:1</td>
<td>20</td>
<td>60</td>
</tr>
<tr>
<td>4:1</td>
<td>12</td>
<td>60</td>
</tr>
<tr>
<td>4:1</td>
<td>12</td>
<td>60</td>
</tr>
<tr>
<td>3:1</td>
<td>15</td>
<td>60</td>
</tr>
</tbody>
</table>

Dutch book: Win positive amount regardless of outcome

\[
P(G) = \frac{1}{1+2} = \frac{1}{3} \quad P(F) = \frac{1}{1+4} = \frac{1}{5} \quad P(B) = \frac{1}{5} \quad P(O) = \frac{1}{1+3} = \frac{1}{4}
\]

\[
P(G \text{ or } F \text{ or } B \text{ or } O) = P(G) + P(F) + P(B) + P(O) = \frac{20}{60} + \frac{12}{60} + \frac{12}{60} + \frac{15}{60} = \frac{59}{60} < 1
\]
A function $P: \mathbb{A} \rightarrow \mathbb{IR}$ satisfying the axioms of probability implies it is not possible to make a "dutch book".

**Theorem**

**Geometric probability**: We choose a point from a region $\Omega \subseteq \mathbb{IR}^2$ in the plane, and each part of the region is "equally likely".

$$P(A) = \frac{\text{area}(A)}{\text{area}(\Omega)}$$

satisfies axioms of probability.
Example:

What is the probability that a randomly selected point is closer to the center than to the boundary?

\[
\text{Probability: } \frac{\left(\frac{r}{2}\right)^2 \pi}{r^2 \pi} = \frac{1}{4} = 25\%
\]

Example: We select a number between \([0, 1]\); \(\Omega\) is the probability that the second digit is at least 7.

\[\Omega = 0.1 \text{ cm}^2\]

\[P(A) = \frac{\text{length}(A)}{\text{length}([0, 1])} = \frac{\text{length}(A)}{1}\]

\[9 \cdot 0.05 = 0.3 = 30\%\]
Family of all "nice" subsets of $\mathbb{S}^2$

Reason: Some sets are not "measurable"  
[We cannot meaningfully define their area or volume]

Banach-Tarski paradox

$B$: Solid ball in $\mathbb{R}^3$, with radius 1

Cut $B$ into $n$ pieces of same size

$P_1, P_2, \ldots, P_n$  

Regroup into

By ball of radius 1

Not "nice":  $\text{Vol}(B) = \sum \text{Vol}(P_i) = \text{Vol}(B_1) + \text{Vol}(B_2) = 2 \text{Vol}(B)$

Volume is not defined